Reply to ''Comment on 'Critical behavior of a traffic flow model' ''

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The use of the dynamical structure factor in order to investigate the transition of the Nagel-Schreckenberg model from free to congested traffic is defended [see the preceding Comment by Chowdhury *et al.*, Phys. Rev. E **61**, 3270 (2000)].

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In our recently published paper [1] we investigated the transition of the Nagel-Schreckenberg model [2] from free flow to jammed traffic. The question whether this transition can be described as a critical phenomenon is controversially discussed in the literature (see, for instance, Ref. [1] and references therein). Phenomenologically, traffic jams correspond to backward moving density waves. Several Ansätze were made to define a single car to be jammed and to extract from such a ''microscopic'' definition an order parameter which describes the transition. However, none of these Ansätze provides a conclusive description of the transition from free to congested traffic flow.

In contrast to these ''microscopic'' Ansätze we therefore used in our work the dynamical structure factor $S(k,\omega)$ which is known to be an appropriate tool to investigate collective behavior. We showed that the dynamical structure factor has the advantage that jams are naturally identified by their negative velocity, i.e., no definition of a single car to be jammed is needed. Analyzing the behavior of the dynamical structure factor for various values of the model parameters (density ρ, noise P) we showed that jams are characterized by a finite correlation length or correlation time below the transition. Approaching the transition these characteristic length and time scales diverge and the corresponding correlation functions decay algebraically. Using a finite-size scaling analysis we showed that in the hydrodynamic limit the smallest jam mode which corresponds to the long range correlations of jams vanishes below the transition. This behavior of the Nagel-Schreckenberg model resembles the behavior of a second-order phase transition where a finite-correlation length diverges at the critical point.

This conclusion was questioned in a Comment by Chowdhury *et al.* [3]. The authors argued that the Nagel-Schreckenberg model does not display criticality at all and that one has to interpret the behavior of the model as a ''crossover type jamming transition.'' Despite some other criticism they listed three arguments to support their view.

(i) First they argued that the fact that the spatial density-density correlation function $C(r)$ does not show an algebraic decay for nonzero noise parameter P is an evidence for a ''crossover.'' But as already pointed out in our paper, to analyze the behavior of the system it is necessary to incorporate both the spatial and temporal evolution of the system. Since the steady state correlation function $C(r)$ and the related steady state structure factor $S(k)$ are time (frequency) averaged they are not an appropriate tool to investigate the transition.

Consider for instance the dynamical structure factor $S(k,\omega)$ which displays in the jammed regime for any given value of k two peaks which correspond to the free flow and the jammed phase, respectively (see Ref. [1]). Integrating over ω one gets the steady state structure factor $S(k)$ where now the information of both different phases is mixed and possible indications of a critical behavior are smeared out. Thus, instead of the steady state functions $S(k)$ [or $C(r)$] it is more appropriate to consider the dynamical structure factor $S(k,\omega)$ [or the related correlation function $C(r,t)$] which contains the necessary information about the spatial and the temporal evolution of the system.

Note that the limit $P=0$ (no noise) is special because in this case the evolution of the system is deterministic, i.e., a snapshot of the system for a given time contains all the information and no analysis of the temporal evolution is needed to describe the system.

(ii) A second argument of Ref. [3] against criticality is that a conventional finite-size scaling analysis of the values of the relaxation time τ as a function of the global density ρ fails in the sense that the maxima of the function $\tau(\rho)$ scale with the system size L but the width does not. According to the definition of the relaxation time [4]

$$\tau=\int_0^\infty [\min\{v^\star(t),\langle v(\infty)\rangle\}-\langle v(t)\rangle]dt \qquad (1)$$

one sums the deviations of the global velocity $\langle v(t)\rangle$ from the minimum of the velocity for zero-density $v^\star(t)$ and the steady state velocity $\langle v(\infty)\rangle$, respectively. Here one has to assume that $\langle v(t)\rangle \leqslant \langle v(\infty)\rangle$ for all t which is fulfilled in the free flow phase. But as Eisenblätter showed [5] this is not fulfilled in the jammed regime where the global velocity exhibits a fast increase to a maximum with $\langle v(t)\rangle > \langle v(\infty)\rangle$ and then it relaxes to the value $\langle v(\infty)\rangle$ (see also Ref. [6]). This behavior yields a negative contribution to the relaxation time and its value is therefore underestimated. Due to this underestimation above the transition the relaxation time is not a symmetric function with respect to the transition density ρ_c and finite-size scaling does not work. For high densities one could even find negative values of the relaxation time. But this negative relaxation time and the failure of

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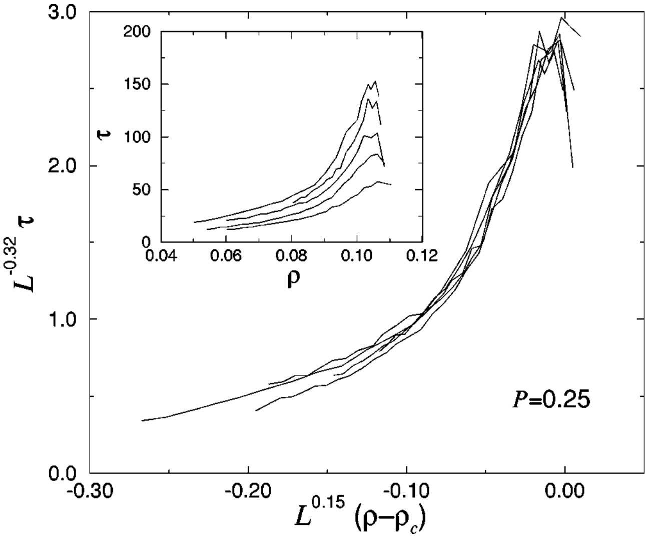

FIG. 1. Finite-size scaling analysis of the relaxation time τ. The inset shows the original values of τ for system sizes $L\in\{2000,\ 6000,\ 10\,000,\ 20\,000,\ 30\,000\}$ (from below to top) obtained from Ref. [5] (see also Ref. [6]). The main figure shows the collapse of the rescaled data according to the usual finite-size scaling Ansätz $\tau(L)=L^x f[L^y(\rho-\rho_c)]$.

finite-size scaling of τ is just a conspicuous result of the unappropriate definition of τ above the transition. It is worth noting that finite-size scaling works below the transition where no negative contributions to τ occur. We plot in Fig. 1 a finite-size scaling analysis of the relaxation time. A good data collapse is obtained below the transition.

The fact that finite-size scaling does not work for all values of the density is just a consequence of the definition of the relaxation time [Eq. (1)] and should not be used to argue against criticality in the Nagel-Schreckenberg model. We think that a refined definition of the relaxation time could prevent from these problems, for instance, the definition

$$\tau_q=\left(\int_0^\infty\left|\min\{v^\star(t),\langle v(\infty)\rangle\}-\langle v(t)\rangle\right|^q dt\right)^{1/q} \qquad (2)$$

with $q=1$ or $q=2$.

(iii) Another argument of Ref. [3] against criticality in the Nagel-Schreckenberg model is connected to the cutoff behavior of the probability distribution of the jam lifetimes [7]. In the ''microscopic'' definition of jams in this work all cars which have a velocity lower than $v_{\max}$ before the randomization update step is applied are defined as jammed. Adjacent jammed cars form clusters and the lifetime of these clusters is examined. Following Ref. [7] the lifetime distribution displays a power-law behavior already in the free flow regime whereas the congested regime is characterized by two different power-law regions. Thus it was concluded that the change from the free flow to the congested regime is accompanied by the appearance of the second power-law regime. It was pointed out in Ref. [7] (and this is the key argument) that the jam lifetime distribution displays a cutoff at $t_c\approx 5\times10^5$, independent of the system size L. If this would be true then the jams exhibit a characteristic finite time scale t_c

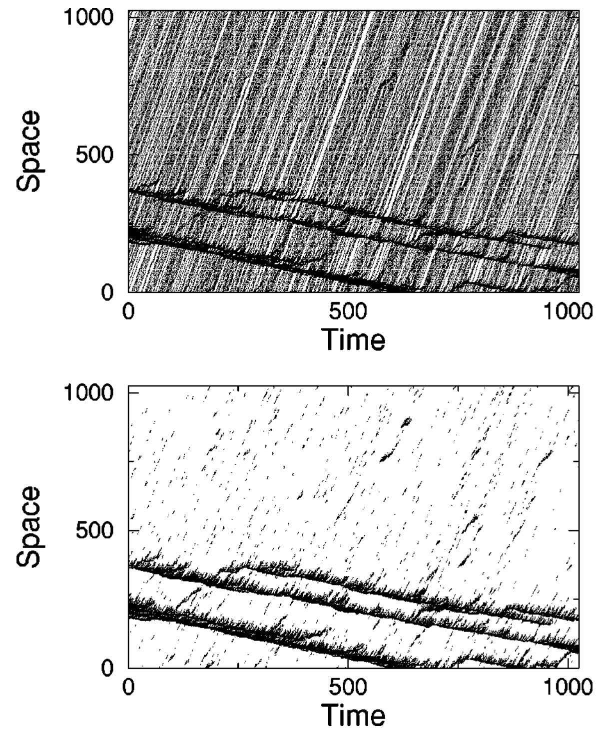

FIG. 2. The space-time diagram of the system. Regions of congested traffic occur as backward moving density waves (upper). Below we display the same car configuration but plotted only cars which are jammed according to the definition of Ref. [7].

which depends only on ρ and P. This finite characteristic time scale, of course, would exclude the existence of criticality at all.

In Fig. 2 we plot a snapshot of the system in a so-called space-time diagram. One clearly sees regions of different densities which correspond to the free flow and jammed phase, respectively. Especially jams are characterized by backward moving density waves. In the lower diagram of Fig. 2 we present the same car configuration but only cars are marked which are jammed according to the definition of Ref. [7]. Surprisingly, clusters with forward moving density waves exist. This is caused by the dynamic rules of the system. The particle interaction (gap condition) can lead to slowing down events of cars which do not coincide with backward moving density waves, i.e., which have nothing to do with real jams. Thus the ''microscopic'' jammed-car definition of Nagel does not distinguish between forward and backward moving density waves at all. Therefore one should be very careful to interpret the results of the corresponding lifetime distributions.

Despite the above mentioned insufficiency of the definition of jammed cars we reinvestigate in the following the corresponding cluster statistics. In Fig. 3 we present the probability distribution of jam lifetimes for various system sizes L. In order to check if the system has reached the steady state we applied $T_{\mathrm{eq}}=10^7$ and $T_{\mathrm{eq}}=10^8$ update steps before we start the actual measurements. Since we found no significant difference between both times we conclude that the steady state has been reached (note that a significantly lower statistic was used in Ref. [7]). In contrast to Fig. 5 in Ref. [7] we find within our simulations that the lifetime distribution displays a weak but significant system size depen-

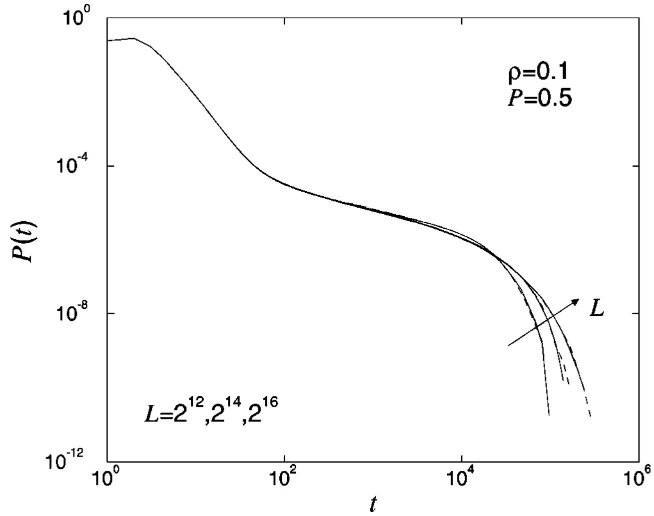

FIG. 3. The lifetime probability distribution of jam clusters for various system sizes and $v_{\mathrm{max}}=5$. In the simulations we used 10^7 (solid lines) and 10^8 (dashed lines) update steps to reach the steady state. The distributions are obtained from an average over 5×10^7 update steps.

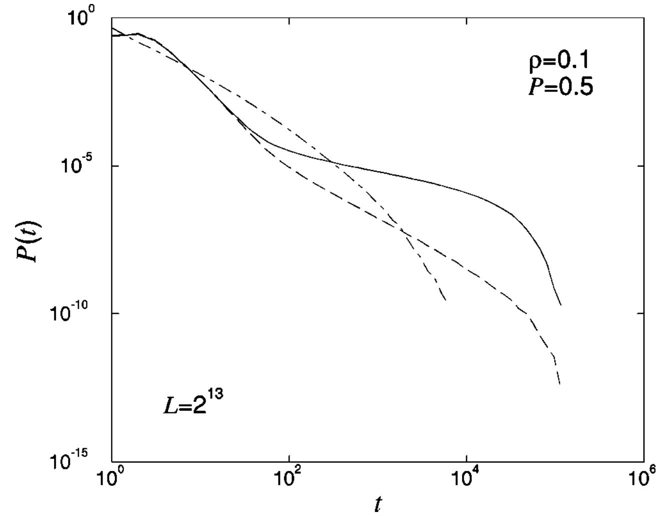

FIG. 4. The probability distribution of jam lifetimes using the original (solid line) and modified (dashed line) cluster labeling (see text for details of the definition) for $v_{\mathrm{max}}=5$. The dot-dashed line corresponds to the different definition of jammed cars according to Vilar and Souza [8].

dence (see Fig. 3) which cannot be neglected in the interpretation of the lifetime distribution. Thus we have no convincing hint that the Nagel-Schreckenberg model exhibits a finite time scale t_c as proposed in Ref. [7]. Additionally we are not aware of any kind of physical effect which could cause such a finite time (or length) scale.

Finally, one can ask how the lifetime distribution depends on the definition of jammed cars and the used method of cluster labeling. In Ref. [7] merging clusters get the same cluster label. The younger cluster ceases but is counted for the statistics. An alternative Ansätz is to avoid this multiple counting of cluster parts. In Fig. 4 we compare the different cluster labeling schemes. Obviously, the long time behavior of the lifetime distribution is different. Additionally we changed the definition of jammed cars and used the definition of Vilar and Souza [8]. In contrast to the assumption of Ref. [3] that the long-time behavior of the lifetime distribution does not depend on the definition of jams we get yet another completely different distribution (see Fig. 4).

In summary, the third argument of Ref. [3] against criticality is based on an insufficient definition of jammed cars which does not distinguish between forward and backward moving density waves. The considered cluster distributions depend strongly on the definition of jammed cars and the used cluster labeling schemes. Scrutinizing the dependence of the lifetime distribution on the system size one cannot neglect finite-size effects.

We are convinced that the problems with the interpretation of the Nagel-Schreckenberg model are caused by the usually considered ''microscopic'' Ansätzes for the jammed cars. It is simply not possible to decide if a single car is jammed just by looking at its velocity or its distance to the forward neighbor. A jam is a collective phenomenon which is characterized by a macroscopic backward moving density wave and an appropriate quantity to examine this collective behavior is the dynamical structure factor. Since no convincing evidence is known that jams are characterized by a finite length or time scale we propose that the performed finite-size scaling analysis of the dynamical structure factor suggests criticality.

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